THE SCOTS COLLEGE

MATHEMATICS EXTENSION I

YEAR 12 ASSESSMENT TASK 3

17TH JUNE 2013

GENERAL INSTRUCTIONS

- Reading time – 2 minutes
- Working time – 45 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- Show all necessary working in Section II

WEIGHTING

20%

TOTAL MARKS

26

QUESTION 1 – 4 (MARKS AS INDICTAED)

- Answers to be recorded in the answer booklets provided.
- *Each question must be completed in a new answer booklet.*
- Label each answer booklet with your name and teachers name and the question number attempted.
- Clearly indicate the booklet order if more than one booklet is used for a question. Eg) Book 1 of 2 and 2 of 2.

QUESTION 1 SEQUENCES, SERIES AND PROBABILITY 5 MARKS

Adam plays a game of dice for a single player with two, 6-sided dice, numbers on each face with the numerals 1 to 6. He wins if the sum of the two upper most faces on each dice after a role is 4, 6 or 8. Conversely, if he roles a sum of 2, 3, 5 or 7 he records a loss. Any other sum of the two dice permits him to continue playing until he records a win or a loss.

a) Find the probability that Adam wins on his first throw of the dice; **1 mark**

b) Show that the probability of Adam throwing the dice again is $\frac{5}{18}$; **1 mark**

c) Show that the probability that Adam wins on the second throw is $\frac{5}{18} \times \frac{13}{36}$; **1 mark**

d) Find the probability that Adam wins the game eventually. **2 marks**

QUESTION 2 SEQUENCES AND SERIES 8 MARKS

a) A spherical bubble is expanding so that its volume is increasing at $10\ \mathrm{mm}^3\mathrm{s}^{-1}$. At what rate is the radius increasing when the surface area is $500\ \mathrm{mm}^2$? **3 marks**

b) The rate at which a body warms in air is proportional to the difference between its temperature B and the constant temperature B_o of the surrounding air, i.e.,

$$\frac{dB}{dt} = k(B - B_o)$$

Where *t* is the time in minutes and *k* is a constant.

i) Show that $B = B_o + Ae^{kt}$, where A is a constant, is a solution of **1 mark**

$$\frac{dB}{dt} = k(B - B_o)$$

ii) For a particular body, when $t = 0, B = 5°\mathrm{C}$ and when $t = 20$, $B = 15°\mathrm{C}$. Given $B_o = 25°\mathrm{C}$, find the temperature of the body after a further 30 minutes have elapsed. Give your answer to the nearest degree. **3 marks**

iii) Explain the behaviour of B as t becomes large. **1 mark**

QUESTION 3 MATHEMATICAL INDUCTION 4 MARKS

Prove by Mathematical Induction that $3n^3 + 6n$ is divisible by 9 for all integers $n \geq 1$. **4 marks**

QUESTION 4 APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD 9 MARKS

An amount $\$A$ is borrowed at r% per annum reducible interest, calculated monthly. The loan is to be repaid in equal monthly instalments of $\$M$.

Let $R = \left(1 + \frac{r}{1200}\right)$ and let $\$B_n$ be the amount owing after n monthly repayments have been made.

a) Show that $B_1 = AR - M$ **1 mark**

b) Show that $B_3 = AR^3 - M(R^2 + R + 1)$ **1 mark**

c) Show that $B_n = AR^n - M\left(\frac{R^n - 1}{R - 1}\right)$ **2 marks**

Julie borrows $900 000 at 6% per annum reducible interest, calculated monthly. The loan is to be repaid in 360 equal monthly instalments.

d) Using the information provided in Parts a to c, show that the monthly instalment should be $5395.95 **2 marks**

e) With the 60th repayment, Julie pays an additional $120 000 so this payment is $125 395.95. After this, repayments continue at $5395.95 per month. How many more repayments will be needed to pay off the loan? **3 marks**

END OF EXAM.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

MATHEMATICS EXTENSION 1.

YEAR 12 ASSESSMENT TASK 3.

17th JUNE 2013

1)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

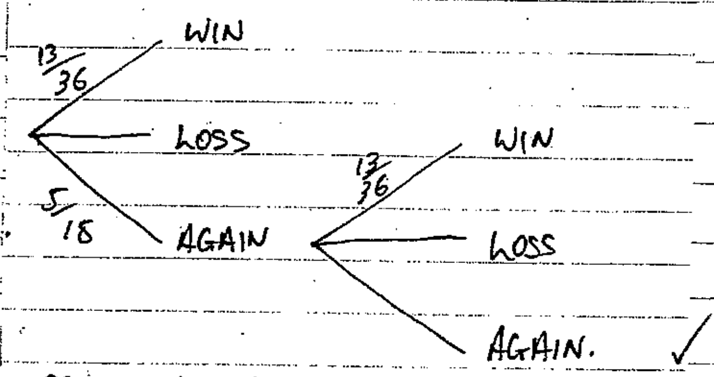
a). $P(\text{win}) = 13/36$ ✓

b). $P(\text{throwing again}) = P(9, 10, 11 \text{ or } 12)$

$= 10/36$

$= 5/18$ ✓

c). 1st Role 2nd Role.



✓

$\therefore P(\text{win } 2^{nd} \text{ role}) = P(\text{AGAIN}) \text{ and } P(\text{WIN})$

$= \frac{5}{18} \times \frac{13}{36}$

d. P(win eventually = P(win 1st role) + ...

... + P(Win 2nd role) + P(win 3rd role) + ...

... + P(win n^{th} role)

$= \frac{13}{36} + \left(\frac{5}{18} \times \frac{13}{36}\right) + \left(\frac{5}{18} \times \frac{5}{18} \times \frac{13}{36}\right) + \left(\frac{5}{18} \times \frac{5}{18} \times \frac{5}{18} \times \frac{13}{36}\right) +$

$+ \ldots + \left(\frac{5}{18}\right)^n \times \frac{13}{36}$

$= \frac{13}{36}\left(1 + \frac{5}{18} + \left(\frac{5}{18}\right)^2 + \left(\frac{5}{18}\right)^3 + \ldots + \left(\frac{5}{18}\right)^n\right)$

$= \frac{13}{36}\left(\frac{1}{1 - 5/18}\right)$ ✓

$= \frac{13}{36} \times \frac{18}{13}$

$= 1/2$ ✓

Question 2.

a). $\frac{dV}{dt} = 10 \text{ mm}^3\text{s}^{-1}$ $SA = 500 = 4\pi r^2$

$$\boxed{\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}}$$

Now:

$V = \frac{4}{3}\pi r^3$ when $SA = 500$

$\therefore \frac{dV}{dr} = 4\pi r^2$ $r^2 = \frac{500}{4\pi} = \frac{1\ldots}{\pi}$

$\therefore 10 = 4\pi r^2 \cdot \frac{dr}{dt}$

$\therefore 10 = 4\pi \times \frac{125}{\pi} \times \frac{dr}{dt}$

$10 = 500 \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = 0.02 \text{ mms}^{-1}$

bi). Given $B = B_0 + Ae^{kt}$

$\frac{dB}{dt} = kAe^{kt}$ where $B - B_0 = Ae^{kt}$

$\therefore \frac{dB}{dt} = k(B - B_0)$ ✓

Hence $B = B_0 + Ae^{kt}$ is a solution of $\frac{dB}{dt} = k(B - B_0)$

Q1. PAS Q2. DAW Q3 MTA Q4 SCD.

bii). When $t=0$, $B=5$

$\therefore 5 = 25 + Ae^0$

$\therefore A = -20$

So, $B = 25 - 20e^{kt}$ ✓

When $t=20$, $B=15$

$\therefore 15 = 25 - 20e^{20k}$

$20e^{20k} = 10$

$e^{20k} = \frac{1}{2}$

$20k = \log_e \frac{1}{2}$

$k = \frac{1}{20} \log_e \frac{1}{2}$

$k = -0.034657$

Then $B = 25 - 20e^{-0.034657t}$ ✓

After a further 30 minutes i.e. $t=50$, ✓

$B = 25 - 20e^{-0.034657 \times 50}$ ✓

$B = 21^\circ$ (to the nearest degree).

biii). A t becomes larger, the term $e^{kt} \to 0$ $(k<0)$ and B approaches the limiting value given by $B = 25^\circ C$, i.e. the body acquires the temperature of the surrounding air. ✓

Question 3

a). LHS $S_n = (1\times4) + (2\times5) + (3\times6) + \dots + n(n+3)$

RTP $S_n = \frac{n}{3}(n+1)(n+5)$

For $n=1$

$LHS = 1 \times 4 = 4$ $\quad$ $RHS = \frac{1}{3}(1+1)(1+5) = \frac{1}{3}(2)(6) = 4$ ✓

So the statement is true for $n=1$. Assume that it is true for $n=k$.

$\therefore S_k = \frac{k}{3}(k+1)(k+5)$

add the next term ✓

$T_{k+1} = (k+1)[(k+1)+3]$

$= (k+1)(k+4)$

So, $S_{k+1} = S_k + T_{k+1}$

$= \frac{k}{3}(k+1)(k+5) + (k+1)(k+4)$

$= (k+1)\left[\frac{k}{3}(k+5) + (k+4)\right]$

$= (k+1)\left[\frac{k^2}{3} + \frac{5k}{3} + \frac{3k}{3} + \frac{12}{3}\right]$

$= \frac{k+1}{3}(k^2 + 8k + 12)$

$= \frac{k+1}{3}(k+2)(k+6)$

$= \frac{k+1}{3}[(k+1)+1][(k+1)+5]$ ✓

Hence, the statement is true for $n=k+1$. By the principle of Mathematical Induction it is true for all $n \geq 1$. ✓

b). $S_n = 3n^3 + 6n$

RTP S_n is divisible by 9.

for $n=1$

$S_1 = 3(1)^3 + 6(1)$ ✓

$= 3 + 6$

$= 9$

$\therefore S_1$ is divisible by 9

Assume that S_n is divisible by 9 when $n=k$;

$\therefore S_k = 3k^3 + 6k = 9M$ ✓

$S_{k+1} = 3(k+1)^3 + 6(k+1)$

$= 3(k^3 + 3k^2 + 3k + 1) + 6k + 6$

$= 3k^3 + 9k^2 + 9k + 3 + 6k + 6$

$= 3k^3 + 9k^2 + 9k + 6k + 9$

$= 3k^3 + 6k + 9k^2 + 9k + 9$

$= 3k^3 + 6k + 9(k^2 + k + 1)$

$= 9M + 9(k^2 + k + 1)$

$\therefore S_{k+1}$ is divisible by 7. Hence, by the Principle of Mathematical Induction the statement is true ~~of~~ for all $n \geq 1$.

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Question 4.

a). $B_1 = A \times \left(1 \times \frac{r}{100} \times \frac{1}{12}\right) - M$

$B_1 = A \times \left(1 \times \frac{r}{1200}\right) - M$ ✓

$B_1 = AR - M$

b). $B_2 = B_1 \times R - M$

$B_2 = (AR - M)R - M$

$B_2 = AR^2 - MR - M$

$B_3 = B_2 \times R - M$

$B_3 = (AR^2 - MR - M)R - M$

$B_3 = AR^3 - MR^2 - MR - M$

$B_3 = AR^3 - M(R^2 + R + 1)$

c). $\therefore B_n = AR^n - M(1 + R + R^2 + R^3 + \dots + R^{n-1})$

$B_n = AR^n - M$ (sum of a GP)

in GP, common ratio $R > 1$ ✓

$\therefore B_n = AR^n - M\left(\frac{R^n - 1}{R - 1}\right)$

d). $B_{360} = 0$

$0 = 900000(R^{360}) - M\left(\frac{R^{360} - 1}{R - 1}\right)$

where $R = \left(1 + \frac{r}{1200}\right)$

$R = \left(1 + \frac{6}{1200}\right)$

$R = 1.005$

$\therefore 0 = 900\,000(1.005^{360}) - M\left(\frac{1.005^{360} - 1}{1.005 - 1}\right)$

$M\left(\frac{5.022575212\ldots}{0.005}\right) = 5\,420\,317.691\ldots$

$M(1004.515042\ldots) = 5\,420\,317.691\ldots$

$M = \$5395.95$

e). Amount owing after 60 payments is $B_{60} - 120\,000$

$B_{60} = 900000(1.005)^{60} - 5395.95\left(\frac{1.005^{60} - 1}{0.005}\right)$

$= 1\,213\,965.137\ldots - 5395.95(69.77003\ldots)$

$= 1\,213\,965.137 - 376\,475.5961$

$= \$837\,489.54$

$\therefore$ Amount owing after 60 payments is:

$837\,489.54 - 120000 = \$717\,489.5412$

If payments continue after this at \$5395.95 per month:

$A = \$717\,489.54$

$M = \$5395.95$

and when the loan is repaid

$B_n = 0$

$\therefore 717\,489.54(1.005)^n - 5395.95\left(\frac{1.005^n - 1}{0.005}\right) = 0$

$717489.54(1.005)^n - \frac{5395.95(1.005)^n}{0.005} + \frac{5395.95}{0.005} = 0$

$(1.005)^n = -\left(\frac{5395.95}{0.005}\right) \div \left(717489.54 - \frac{5395.95}{0.005}\right)$

$(1.005)^n = 2.983656698\ldots$

$n \log_e 1.005 = \log_e 2.983656698\ldots$

$n = \frac{\log_e 2.983656698\ldots}{\log_e 1.005}$

$n = 219.18$

There will be at most 220 further payments needed.

$n = 219.1760461\ldots$

$n \doteq 219$ (accept this answer).